Local-Global Phenomena and Applications to Zaremba's Conjecture and Circle Packings

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Continued Fractions

Given $\alpha \in [0, 1]$. To create a continued fraction for α:

1. Let $a_1 = \lfloor \alpha \rfloor$.
2. Form $\alpha_1 = \frac{1}{\alpha} - a_1$. Notice that $\alpha_1 \in [0, 1]$.
3. If $\alpha_1 \neq 0$ let $a_1 = \lfloor \frac{1}{\alpha_1} \rfloor$. Otherwise terminate this process.
4. Form $\alpha_2 = \frac{1}{\alpha_1} - a_2$
5. Repeat the process indefinitely or until termination.

Then

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Each a_i is called a **partial quotient**, and $\alpha = [a_1, ...]$.

Examples of Continued Fractions

The golden ratio

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

A rational number

$$\frac{2}{5} = \cfrac{1}{2 + \cfrac{1}{2 + 0}}$$

Zaremba's Conjecture

Conjecture

There exists A such that for all $q \in \mathbb{N}$ there exists $a \in \mathbb{Z}$ with $(a, q) = 1$ and $\frac{a}{q}$ has partial quotients bounded by A.

Motivation: Quasi-Monte Carlo methods for selecting optimal sampling points for multi-dimensional numerical integration of multivariate integrands.

Failures of Zaremba's Conjecture for Specific Values of A and q

Consider $A = 4$ and $q = 6$.

Then the only possible values for a as the numerator with reduced fraction $\frac{a}{q}$ are $a = 1$ and $a = 5$.

However, $\frac{1}{6} = [6]$ and $\frac{5}{6} = [1, 5]$. Thus, there are no values a for which the partial quotients of $\frac{a}{q}$ are bounded by A.

Matrix Connection to Continued Fractions

Consider $\alpha = [a_1, a_2]$ and the matrices

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix}$$

Their product is as follows

$$M_1 M_2 = \begin{bmatrix} 1 & a_2 \\ a_1 & 1 + a_1 a_2 \end{bmatrix}$$

Naturally one may identify this matrix in the following way

$$\begin{bmatrix} 1 & a_2 \\ a_1 & 1 + a_1 a_2 \end{bmatrix} \mapsto \frac{a_2}{a_1 a_2 + 1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + 0}}$$

In fact, if $\alpha = \frac{b}{d} = [a_1, ..., a_n]$

$$\begin{bmatrix} * & b \\ * & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}$$

Relation to Zaremba's Conjecture

For some A as given in Zaremba's Conjecture, let the family of matrices

$$\eta = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix} \right\}_{i=1}^{A}$$

And form the semi-group with the operation of matrix multiplication

$$\Gamma = \langle \eta \rangle^{+}$$

Then consider the projection mapping taking a matrix $M \in \Gamma$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto d$$

Essentially, this "pulls out the denominator" formed by the continued fraction that M represents.

Local-Global Phenomena

Zaremba's Question: What is the image in $\mathbb{Z}$ of the projection $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto d$?

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Answer: In the case laid out above, for $A = 2$ the image of the projection modulo n is all of $\{0, ..., n-1\}$. Thus, in this case, there are no **congruence restrictions**.

However, this is not always the case. Given a particular semi-group Γ and a projection $f : \Gamma \to \mathbb{Z}$ there may be congruence restrictions upon the image $f(\Gamma)$ (i.e. no elements congruent to 7 modulo 13 exist in the image).

Local-Global Phenomena (Continued)

Consider a semi-group Γ and a projection $f : \Gamma \to \mathbb{Z}$.

Local-Global Conjecture If $n \in \mathbb{N}$ is sufficiently large and does not fail any congruence restrictions (n is called "admissible") then n is in the image of Γ under f.

Project Outline

Explore problems similar to Zaremba's Conjecture and circle/sphere packings focusing on local-global phenomena.

- Test where the Local-Global Conjecture fails and where it works.
- Examine fibers of the projections selected.
- Create simulations for various Γ and f.

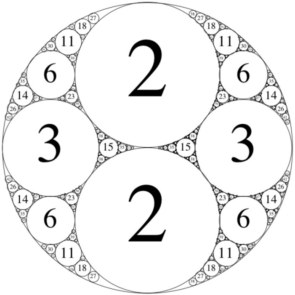

Image courtesy of Dr. Kontorovich

Phases of Project

1. Collect literature sources.
2. Gain requisite background knowledge.
3. Simulate various problems.
4. Create and prove conjectures based upon simulations.